Lecture 35

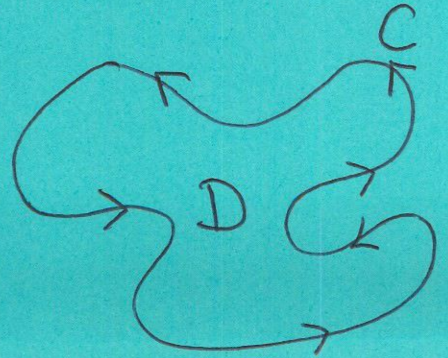
16.4 - Green's Theorem

To deal with Green's theorem, we need certain kinds of curves.

<u>Def</u>: A curve C is <u>simple</u> if it does not cross itself between the endpoints, i.e., if $\vec{r}(t)$, $a \le t \le b$, parametrizes C, then $\vec{r}(t_1) \neq \vec{r}(t_2)$ for any $a < t_1 < t_2 < b$.

<u>Def</u>: A curve C (in the plane) is called <u>positively oriented</u> if it is traversed counterclockwise.
If C bounds a region D, then we say C has positive orientation if when you traverse C, the region D is on your left.

<u>Ex</u>:



In this situation, we write $C = \partial D$.

Green's Theorem

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane which bounds a region D. If P and Q have continuous first partials on a region containing D, then

$$\boxed{\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}.$$

<u>Notation</u>: For a closed curve C, we write

$$\oint_C P\,dx + Q\,dy$$

to denote the line integral along C where C has the positive orientation.

If we rewrite C as ∂D, then Green's theorem reads:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\partial D} P\,dx + Q\,dy$$

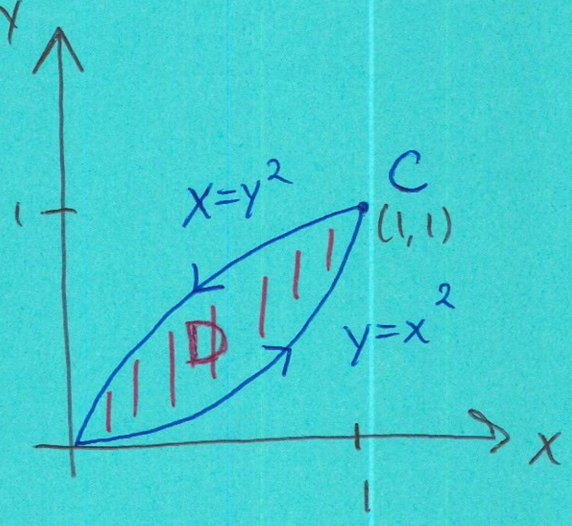
Again, we see this theme of swapping a derivative for a boundary. In this respect, Green's theorem is a version of the fundamental theorem of calculus for double integrals.

Ex: Compute the line integral

$$\oint_C (y+e^{\sqrt{x}})\,dx + (2x+\cos y^2)\,dy$$

where C is the boundary of the region bounded by $y=x^2$ and $x=y^2$

Sol: The region is



By Green's theorem,

$$\oint_C (y+e^{\sqrt{x}})\,dx + (2x+\cos y^2)\,dy$$

$$= \iint_D \left(\frac{\partial}{\partial x}(2x+\cos y^2) - \frac{\partial}{\partial y}(y+e^{\sqrt{x}})\right) dA = \iint_D 1\, dA$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} dx\, dy = \int_0^1 (\sqrt{y} - y^2)\,dy = \left(\frac{2}{3}y^{3/2} - \frac{1}{3}y^3\right)\Big|_0^1$$

$$= \frac{1}{3}$$ ◇

A useful trick to know involving Green's theorem:

$$\boxed{\text{Area}(D) = \iint_D dA = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx}$$

<u>Ex</u>: Compute $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \dfrac{-y\hat{i} + x\hat{j}}{x^2+y^2}$ and C is the unit circle $x^2+y^2=1$.

<u>Sol</u>: A parametrization of C is $\vec{r}(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. This traverses C counterclockwise, so has the correct orientation. Then $\vec{F}(\vec{r}(t)) = \langle -\sin t, \cos t\rangle$ and $\vec{r}\,'(t) = \langle -\sin t, \cos t\rangle$. So:

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$$

Now, $\oint_C \vec{F}\cdot d\vec{r} = \oint_C \underbrace{\dfrac{-y}{x^2+y^2}}_{P}dx + \underbrace{\dfrac{x}{x^2+y^2}}_{Q}dy$

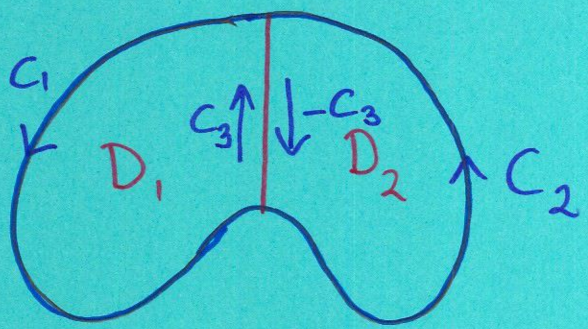
If we use Green's theorem, since $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$ we would have $\oint_C \vec{F}\cdot d\vec{r} = \iint_D 0\,dA = 0$, an apparent contradiction! The problem here is that $\vec{F}$ is not defined at the origin, and also that P & Q are not C^1. Moral: BE CAREFUL! ◇

This leads us to ask: How do we deal with holes? First, let's look at splitting up regions.

If $D = D_1 \cup D_2$, then we know:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Eg.: D:



$\partial D_1 = C_1 \cup C_3$

$\partial D_2 = C_2 \cup (-C_3)$

So, Green's theorem gives

$$\iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{C_1 \cup C_3} P\,dx + Q\,dy = \int_{C_1} P\,dx + Q\,dy + \int_{C_3} P\,dx + Q\,dy$$

$$\iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{C_2 \cup (-C_3)} P\,dx + Q\,dy$$

$$= \int_{C_2} P\,dx + Q\,dy + \int_{-C_3} P\,dx + Q\,dy = \int_{C_2} P\,dx + Q\,dy - \int_{C_3} P\,dx + Q\,dy$$

So,

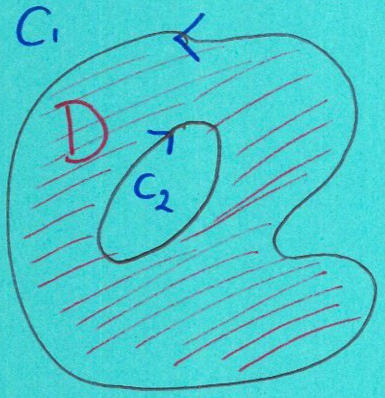
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{C_1} P\,dx + Q\,dy + \cancel{\int_{C_3} P\,dx + Q\,dy} + \int_{C_2} P\,dx + Q\,dy - \cancel{\int_{C_3} P\,dx + Q\,dy}$$

$$= \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy .$$

So, we can integrate over boundaries piece by piece.

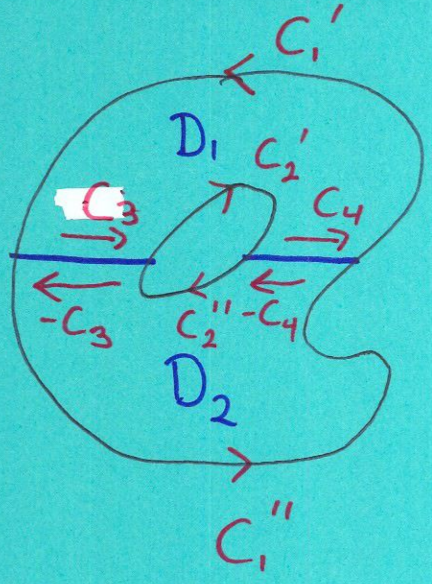
So, what if D looks like:



Here $\partial D =$?

Using our conventions C_1 gets the counterclockwise orientation and C_2 gets the clockwise orientation.

Then, we split D into 2 pieces: D_1 & D_2

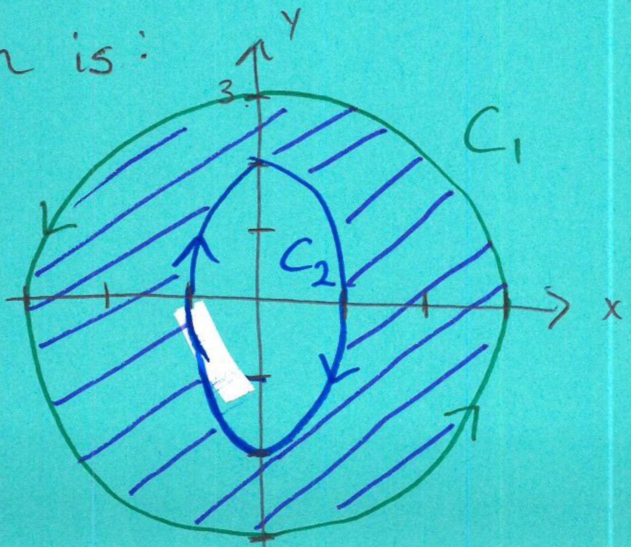


Using the same idea as above, we arrive at

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{C_1} P\,dx + Q\,dy - \oint_{C_2} P\,dx + Q\,dy$$

<u>Ex</u>: Find the area of the region inside the circle of radius 3, centered at the origin, but outside the ellipse $4x^2 + y^2 = 4$.

<u>Sol</u>: Our region is:



We have $\partial D = C_1 \cup (-C_2)$. We wish to compute

$$\text{Area}(D) = \iint_D dA = \oint_{C_1} P\,dx + Q\,dy - \oint_{C_2} P\,dx + Q\,dy$$

If we take $P\,dx + Q\,dy = \frac{1}{2}(-y\,dx + x\,dy)$, then

$$\text{Area}(D) = \frac{1}{2}\left(\oint_{C_1} x\,dy - y\,dx - \oint_{C_2} x\,dy - y\,dx\right)$$

A parametrization of C_1 is: $\vec{r}_1(t) = \langle 3\cos t, 3\sin t\rangle$, $0 \le t \le 2\pi$.

" " " C_2 is: $\vec{r}_2(t) = \langle \cos t, 2\sin t\rangle$, $0 \le t \le 2\pi$.

$$\oint_{C_1} x\,dy - y\,dx = \int_0^{2\pi} \big((3\cos t)(3\cos t\,dt) - (3\sin t)(-3\sin t\,dt)\big)$$

$$= \int_0^{2\pi} 9\,dt = 18\pi$$

$$\oint_{C_2} x\,dy - y\,dx = \int_0^{2\pi} \big((\cos t)(2\cos t\,dt) - (2\sin t)(-\sin t\,dt)\big)$$

$$= \int_0^{2\pi} 2\,dt = 4\pi$$

So, $\text{Area}(D) = \frac{1}{2}\left(\oint_{C_1} x\,dy - y\,dx - \oint_{C_2} x\,dy - y\,dx\right) = \frac{1}{2}(18\pi - 4\pi) = 7\pi$

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